APPLIED OPTIMIZATION AND APPROXIMATION

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1. Optimization in kinematics

In the space R^n we consider a system of p material particles, of masses $m_1,...,m_p$, which are moving following the action of a the continuous fields of forces

$$\vec{f}_1,..., \vec{f}_p, \ \vec{f}_l = \vec{f}_l(t), t \in [t_0, T], l = 1,..., p.$$

The velocities $\vec{v}_1,...,\vec{v}_p$ verify the following conditions

$$\vec{v}_l(t_0) = \vec{0}, \vec{v}_l \in C^1([t_0, T], R^n), l = 1, \ldots, p.$$

1.1. Our first aim is determining conditions on the velocity-fields**,** such that the kinetic energy and the total kinetic moment of the system to be maximal**.** We assume that the euclidean norms in R^n of the forces fields are bounded from above by the constants $M_l, l = 1, \ldots, p,$ so that by hypothesis we have:

$$\left\| \vec{f}_l(t) \right\| \leq M_l, \forall t \in [t_0, T], l = 1, \ldots, p.$$

Theorem 1.1. (i) *If the movement of the system occurs such that the kinetic energy is maximal, the* $\vec{f}_l$ *is parallel to* $\vec{v}_l$ *at any moment of time, and we have*

$$v_{l,\max}(t) = (M_l / m_l) \cdot (t - t_0), \ l = 1, \ldots, p,$$

$$T_{\max}(t) = \frac{1}{2}\left(\sum_{l=1}^{p}\left(M_l^2 / m_l\right)\right) \cdot (t - t_0)^2, t \in [t_0, T].$$

(ii) *Under the additional assumption of a maximal kinetic momentum* $\mathbf{M} = \mathbf{M}(t)$*, we have:*

$$v_1(t) = \cdots = v_l(t) = v(t), t \in [t_0, T],$$

$$M_1 / m_1 = \cdots = M_p / m_p,$$

$$\mathbf{M}_{\max}(t) = \left(\sum_{l=1}^{p} m_l \right) (M_k / m_k)(t - t_0), k = 1, \ldots, l.$$

We next consider a method of increasing the "total momentum", illustrated in a particular case of a train formed by a locomotive and a wagon of masses m_1, m_2 respectively. The locomotive is coupled to the wagon with the aid of an elastic connection, of coefficient $k > 0$. Let F be the force applied to the locomotive and μ a friction coefficient. At the initial moment $t = t_0$, the velocities are assumed to be zero. We assume that the movement is rectilinear. From Newton's second low, the movement equations of the train follow:

$$m_1\ddot{x}_1 = F - k(x_1 - x_2) - \mu\, g\, m_1\dot{x}_1;$$
$$m_2\ddot{x}_2 = k(x_1 - x_2) - \mu\, g\, m_2\dot{x}_2,$$

where x_1, x_2 are respectively the abscissa of the gravity centers of the two vehicles.

Theorem 1.2. *The momentum* $y = m_1\dot{x}_1 + m_2\dot{x}_2$ *is maximal if and only if*

$$\dot{x}_1(t) = \dot{x}_2(t),\ y_{\max}(t) =$$
$$(m_1 + m_2)\dot{x}_1(t) = (m_1 + m_2)\dot{x}_2(t) =$$
$$(m_1 + m_2)^{1/2}(2T(t))^{1/2}, T_{optimal} = \frac{y_{\max}^2(t)}{2(m_1 + m_2)} =$$
$$\frac{F_{\max}^2[1 - \exp(-\mu g(t - t_0))]^2}{2(m_1 + m_2)\mu^2 g^2} \uparrow \frac{F_{\max}^2}{2(m_1 + m_2)\mu^2 g^2}, t \to \infty,$$

where the force $F_{\max}$ *is assumed constant with respect to* t. *It follows that the kinetic energy corresponding to the maximal momentum is increasing with* t, *having a horizontal asymptote at* ∞.

2. Minimizing the cost function

Next we go on with a **c**onstrained minimization problem of the total cost, which follows from the mean inequality. We recall this problem due to its simplicity and importance in applications. Let consider the constrained minimization problem

$$\min\left(\sum_{j=1}^{n} c_j x_j\right), c_j > 0, x_j \geq 0,\ j = 1,\dots,n,$$

$$\prod_{j=1}^{n} x_j := P_2 = const.$$

Application of the mean inequality lieds to the following result.

Theorem 2.1. *The minimum of the total cost, constrained by condition* $\prod_{j=1}^{n} x_j = P_2 = const.$ *is realized for equal values of the products* $c_j x_j$, $j = 1,\ldots,n$. *Namely, we have*

$$x_{j,opt.} = \left(\sqrt[n]{P_1 P_2} / c_j\right),\ j = 1,\ldots,n,$$

$$\min\left\{\sum_{j=1}^{n} c_j x_j ;\ \prod_{j=1}^{n} x_j = P_2\right\} = n \cdot \left(\sqrt[n]{P_1 P_2}\right),$$

where $\boldsymbol{P_1 = \prod_{j=1}^{n} c_j}$.

3. Approximation on unbounded subsets and the moment problem

Let $\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_n$, where ν_j, $j = 1, \ldots, n$ are positive Borel regular $M-$ determinate measures on $\mathbb{R}$, with finite moments of all natural orders. Let

$$\varphi_j(t_1, \ldots, t_n) = t_1^{j_1} \cdots t_n^{j_n},$$
$$j = (j_1, \ldots, j_n) \in \mathrm{N}^n, (t_1, \ldots, t_n) \in R^n.$$

Let $X = L^1_\nu(R^n)$, and Y be an order complete Banach lattice with solid norm, and $(y_j)_{j \in \mathrm{N}^n}$ a sequence in Y.

Theorem 3.1. *Let* $F_2 : X \to Y$ *be a positive linear bounded operator. The following statements are equivalent:*

(a) *there exists a unique linear operator* $F : X \to Y$, *such that* $F(\varphi_j) = y_j, \forall j \in \mathrm{N}^n$, F *is between zero and* F_2 *on the positive cone of* X, $\|F\| \leq \|F_2\|$;

(b) *for any finite subsets* $J_k \subset \mathrm{N}, k = 1,\ldots,n$ *and any* $\{\lambda_{j_k}\}_{j_k \in J_k} \subset R, k = 1,\ldots,n,$ *we have:*

$$0 \leq \sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} y_{i_1 + j_1, \ldots, i_n + j_n} \right) \cdots \right) \leq$$

$$\sum_{i_1, j_1 \in J_1,} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} F_2\left(\varphi_{i_1 + j_1, \ldots, i_n + j_n} \right) \right) \cdots \right).$$